Assignment 5.

This homework is due *Tuesday* 11/02/2010.

There are total of 40 points in this assignment. 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

This assignment covers sections 5.1–5.3.

(1) [3pt] (Exercise 5.1.3, Gluing Lemma) Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) = f(x)$ for $x \in [a, b]$ and $h(x) = g(x)$ for $x \in (b, c]$. Prove that h is continuous on $[a, c]$.

(2) (a) [2pt] (Exercise 5.1.5) Let f be defined for all $x \in \mathbb{R}$, $x \neq 2$, by $f(x) = \frac{x^2+x-6}{x-2}$. Can f be defined at $x = 2$ in such a way that f is continuous at this point?

(b) [2pt] Same question about $g(x) = \frac{x^2+x-7}{x-2}$.

(3) (a) [3pt] (Exercise 5.1.12) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Show that $f(x) = 0$ at every point $x \in \mathbb{R}$.

(b) [3pt] (Exercise 5.2.8) Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

(4) [3pt] (Exercise 5.2.2) Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$ for $x \in \mathbb{A}$, is continuous on A.

(5) (Exercise 5.2.3) Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that

(a) [3pt] the sum $f + g$ is continuous at c,

(b) [3pt] the product fg is continuous at c.

(6) [4pt] (Exercise 5.2.5) Let g be defined on $\mathbb{R}$ and by $g(1) = 0$, and $g(x) = 2$ if $x \neq 1$, and let $f(x) = x+1$ for all $x \in \mathbb{R}$. Show that $\lim\limits_{x\to 0} g \circ f \neq (g \circ f)(0)$. Why doesn't this contradict Composition of Continuous Functions Theorem (Theorem 5.2.6)?

— see next page —

(7) (a) [3pt] (Part of exercise 5.3.5) Show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has at least two real roots.

(b) [4pt] (Exercise 5.3.4) Show that every polynomial of odd degree with real coefficients has at least one real root.

(8) (a) [4pt] (Exercise 5.3.11) Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : \ f(x) < 0\}$, and let $w = \sup W$. Prove that $f(w) = 0$. (This provides an alternate proof of Intermediate Value Theorem.)

(b) [3pt] Why the same reasoning does not necessarily work if both $f(a) > 0, f(b) > 0$? (That is, find a precise place in the construction above that doesn't go through in such case.)